NOTES

Never Rush to Be First in Playing Nimbi

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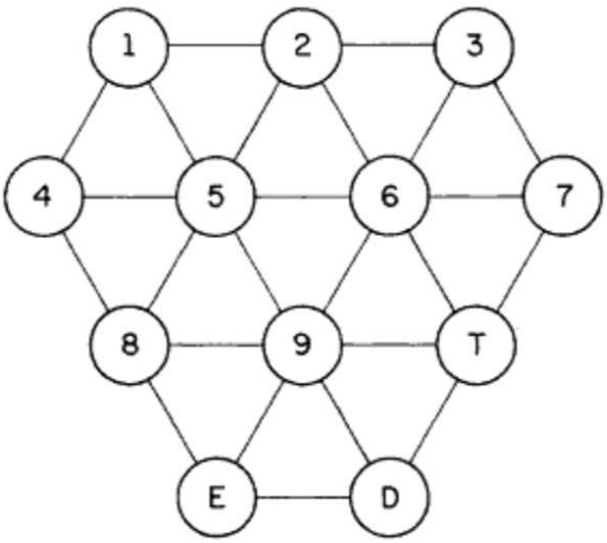
Nimbi is a two-player board game invented by Piet Hein. Twelve tokens, placed on the twelve vertices of a hexagonal board numbered 1,2,...,9,T,E,D, constitute the initial position of Nimbi (see FIGURE 1). The board contains twelve **rows**: four horizontal rows, four diagonal rows in northeast to southwest direction and four diagonal rows in northwest to southeast direction. The two players play alternately. Each player at his turn can remove any contiguous portion from a single row. (Thus 159D may be removed in one move. But if 5 had been removed previously, then neither 19 nor 1D nor 19D can be removed in a single move, though 1 or 9 or D or even 9D can.) In Piet Hein's version, the player making the last move is the loser, his opponent the winner. We call this version LPL-Nimbi. The other version, in which the player making the last move wins and his opponent loses, will be called LPW-Nimbi.



The Nimbi Board

FIGURE 1

Both versions of Nimbi are **combinatorial games**, which, for our purposes, are defined to comprise all finite two-player 0-1 games (finite: finite number of positions; 0-1: outcomes are lose and win only) with perfect information (unlike some card games where information is hidden) and without chance moves (no dice), in which the players play alternately. A combinatorial game is **last player losing** (LPL) if the player first unable to move wins, and it is **last player winning** (LPW) if the player first unable to move loses. In every combinatorial game, either the first or the second player has a winning strategy. A simple proof of this fact, due to Steinhaus, is given by Kac [7]. Since it is very short, we reproduce it here. Denote by $a_1, a_2, \cdots$ and by $b_1, b_2, \cdots$ the moves of Al and Beth, respectively. Suppose that Al makes the first move. The fact that Beth has a winning strategy can be expressed symbolically as follows:

$$(\forall a_1)(\exists b_1)(\forall a_2)(\exists b_2)\cdots(\forall a_n)(\exists b_n) \quad \text{Beth wins.}$$

The negation of this statement is obtained by the familiar De Morgan's rule, and it reads:

$$(\exists a_1)(\forall b_1)(\exists a_2)(\forall b_2)\cdots(\exists a_n)(\forall b_n) \quad \text{Beth does not win.}$$

This, however, is clearly the statement that Al has a winning strategy, and the proof is complete.

Nimbi

Nimbi has a long and dramatic history. It is the last replique in a dialogue down the ages.

Probably the oldest game in the world is Nim, which orginated in the Orient thousands of years ago. It was played with the most simple material: 12 stones usually placed in heaps of 3, 4, and 5. Two players took turns making a move. A move consisted in removing from any one of the heaps as many and as few stones as one wished, i.e., at least one stone and at most the whole heap. The aim was to force the opponent to take the last stone.

Simple in principle but difficult to master.

It has entertained people all over the world for thousands of years and kept them groping for a general principle to reveal the right moves in each situation. In 1901, the French-American mathematician Charles Leonard Bouton, succeeded—by means of a subtle analysis—to find a very simple principle, applicable by anyone, telling you whether a situation was lost or won and in the latter case which move or moves would ensure you the final victory. So the ancient game was turned into a beautiful mathematical solution but was destroyed as a game. This destruction was taken up as a challenge by the Danish author, scientist and inventor, Piet Hein, who set himself the task to revive Nim and give it back its old dignity as an unconquered game. And this by means of a change that should not make it less simple in principle but should bring it outside the reach of the analysis of Charles Leonard Bouton.

Half a century after Bouton's assassination of Nim, Piet Hein succeeded in this strange and difficult task, the greatest difficulty being to save the simplicity in principle, at the same time making it unconquerable by analysis. Piet Hein's new principle was made the topic of an article by Martin Gardner in "Scientific American" and in one of his books about Mathematical Puzzles and Diversions. During a couple of decades mathematicians have tried to destroy even this new game, attempting to find a general principle that would cover all versions of it with varying numbers of stones, as Bouton's analysis did in respect of Nim. Their efforts hitherto have been in vain and there are considered to be fair chances that Piet Hein has succeeded so thoroughly that his game will never be destroyed. For the principle, if ever one is found, is likely to be so complicated that the actual experimenting in the single situations proves to be simpler; meaning, that the playing of it is not beaten by theory, thus forever reassuring its position as a game.

—*from the game brochure*

Since both LPL-Nimbi and LPW-Nimbi are combinatorial games, either the first or the second player must have a winning strategy in each.

The purpose of this note is to prove the following more specific (and perhaps surprising) result.

THEOREM. *In both LPL-Nimbi and LPW-Nimbi, the second player can force a win.*

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NEXT

This theorem is a "rare event" because almost all combinatorial games (whether LPL or LPW) are games in which the first player can force a win (Singmaster [8]). More precisely, if $F(n)$ is the number of combinatorial games of length not exceeding n for which the first player has a winning strategy and $N(n)$ is the total number of combinatorial games of length not exceeding n (where the length of a game is the number of moves from beginning to end), then $\lim_{n\to\infty} F(n)/N(n)=1$.

Before proving the theorem we shall summarize briefly some of the basic concepts and vocabulary of combinatorial games. Let Γ be a combinatorial game, S its set of positions, and a, b two positions in S. Then b is a **follower** of a if there is a move from a to b. A position with no follower is **terminal**, while an initial position (a position without predecessor) is called a **source**. A position q is called an ***N*-position** if the Next player can force a win from q irrespective of the moves of his opponent, or a ***P*-position** if the Previous player can force a win from q irrespective of the moves of his opponent. This dichotomy partitions the set S of all positions into the subset N of N-positions and the subset P of P-positions, where the partition depends on whether Γ is LPL or LPW. It follows that a position is an N-position if and only if it has a follower in P, and it is a P-position if and only if all its followers are in N. It thus appears that P-positions are relatively rare. Singmaster's result is a precise statement of this fact. Our analysis of Nimbi rests on the determination of a certain subset of P-positions.

First, however, we must define a so-called **Sprague-Grundy function** g which maps the set of positions S into the set of nonnegative integers. For each position $a\in S$, the Sprague-Grundy number $g(a)$ is defined to be the smallest nonnegative integer not appearing in the set $\{g(b)\}$ of all followers b of a. Thus, in particular, $g(a)=0$ if a is terminal. The importance of g for combinatorial games stems from the following two facts:

I. $P=\{a\in S: g(a)=0\}$ if Γ is an LPW-game.
II. The g-value of a position in a "disjunctive sum" game is the "nim-sum" of the g-values of the individual positions.

The second fact needs some explanation. Suppose that two players play a game Γ consisting of a finite collection of disjoint combinatorial games $\Gamma_1,\Gamma_2,\ldots,\Gamma_m$, also called components, where each player at his turn selects some component Γ_i and makes a move in it. Then the game Γ is called the **disjunctive sum** of the games $\Gamma_1,\Gamma_2,\ldots,\Gamma_m$. The g-value of a position in Γ is, according to II, the nim-sum of the g-values of the positions in the components Γ_i. To find the nim-sum, write each g-value to the base 2 as $\Sigma a_r 2^r$, then add the a_r's modulo 2 for each value of r without carrying to obtain a binary sum. For example, the nim-sum of 1, 2 and 3 is 0 (since $1_2\oplus 10_2\oplus 11_2=00_2$, where $\oplus$ denotes nim-sum) and the nim-sum of 3 and 6 is 5. The nim-sum of two numbers is 0 if and only if the numbers are the same. (For these and related facts about combinatorial games, see Conway [1], Smith [9] and Fraenkel [3].)

We are now ready to analyze LPL-Nimbi, using a position catalogue (TABLE 1a, b) containing a set of P-positions large enough to prove that the second player in LPL-Nimbi can always force a win. In TABLE 1 a position such as 127T is really just a sample of position number 1, because 23ED and other instances represent the same position. The names in TABLE 1 are designed to help in recognizing the shape of different samples representing the same position.

We have to verify that each position in TABLE 1a, b is a P-position. This is illustrated for position number 19 of TABLE 1b: We summarize in TABLE 2 the 32 possible moves the Next

Position Number	Sample Position	Name
1	127T	two 2's
2	2478TD	1,2, straight 3
3	1378TE	1,2, crooked 3
4	2578TD	two straight 3's
5	13468T	two crooked 3's
6	13479ED	two 2's and triangle
7	4578TED	small kite
8	134578TED	big kite
9	123456789TED	full board

(a)

Position Number	Sample Position	Name
10	1	one dot
11	1TE	three dots
12	19ED	dot and triangle
13	2689	small bucket
14	349TED	two dots and rhombus
15	178TED	dot and sled
16	24679D	dot and horse
17	124679	wrench
18	1234567	span
19	1245679TD	span with two triangles

(b)

Position Number	Sample Position	Name
20	(empty)	void
21	1T	two dots
22	9ED	triangle
23	278D	four dots
24	349ED	two dots and triangle
25	29TED	dot and rhombus
26	1568T	horse
27	12359	letter *A*
28	137TED	dot and big bucket
29	125679TD	span with triangle

(c)

A catalogue of *P*-positions (previous player winning positions) in the game of Nimbi. Those listed in (a) are *P*-positions for both LPL (last player losing) and for LPW (last player winning) versions of the game; those listed in (b) are *P*-positions only for LPL-Nimbi, while those in (c) are *P*-positions only for LPW-Nimbi.

TABLE 1

player can make by listing the locations he vacates (grouped by type), followed by a *P*-position number from TABLE 1a, b which the Previous player can attain by his countermove. Since the Previous player can reach a *P*-position in each of the 32 cases, position number 19 is indeed in *P*. To complete the proof, the reader should make a similar verification for each of the remaining *P*-positions of TABLE 1a, b. The full board is position number 9. To see that it is a *P*-position in LPL-Nimbi, verify that any move from position number 9 can be countered by a move to one of the *P*-positions numbered 2, 7, 8, 18 or 19. This completes the proof that LPL-Nimbi is a second player winning game.

In FIGURE 2 we present a strategy graph whose vertices are *P*-positions of LPL-Nimbi. Two vertices a and b are joined by a downward directed edge (a, b) if for some move of the Next player from a, the Previous player can respond by moving to b. Since all possible moves from

Type	Next Move	Attainable *P*-position	Type	Next Move	Attainable *P*-position	Type	Next Move	Attainable *P*-position
→4	4567	12	↗2's	TD	17	singles	D	1
↖4	159D	12		7T	13		T	16
→3's	567	12		69	14		9	1
	456	1		25	1		7	18
↗3	7TD	1		14	12		6	14
↖3's	59D	12					5	17
	159	11	↖2's	9D	12		4	18
	26T	12		59	13		2	12
→2's	9T	14		15	16		1	12
	67	11		6T	11			
	56	5		26	1			
	45	13						
	12	12						

The 32 possible moves the Next player can make from position number 19 and their rebuttals.

TABLE 2

each P-position are taken into account, the graph gives enough information to enable the second player to win in all cases. Position 9 on top is the source, while position 10 at the bottom is the terminal position of the graph. The vertices of the strategy graph are precisely the P-positions of TABLE 1a, b, except for position 4 which is not used. If the second player adheres to the strategy indicated by this graph, player 1 can delay losing for at most eight moves. A possible sequence of such moves, omitting intervening N-positions, is $9 \to 8 \to 3 \to 1 \to 10$.

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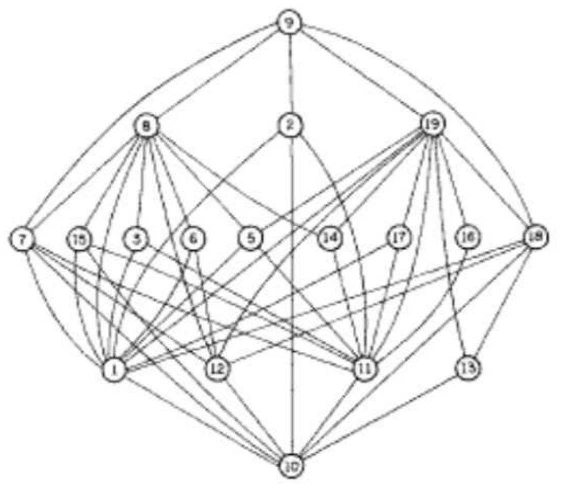
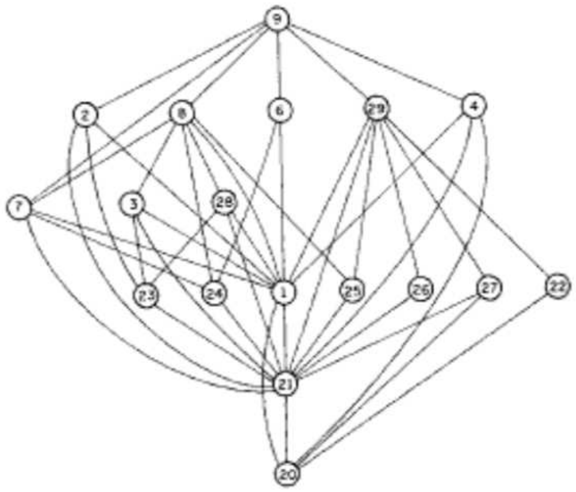
We turn now to LPW-Nimbi. In addition to a direct analysis via P-positions, as we just did for LPL-Nimbi, LPW games can be analyzed by using property I of the g-function. Also, property II can be used for those positions which can be decomposed into a disjunctive sum of smaller positions. It is easy to verify that the P-positions numbered 1 through 8 (but not 10 through 19) of LPL-Nimbi in TABLE 1 have g-value 0 and are, therefore, also P-positions in LPW-Nimbi. A list of ten additional P-positions of LPW-Nimbi (which are not P-positions of



A Strategy Graph for LPL-Nimbi

FIGURE 2



A Strategy Graph for LPW-Nimbi

FIGURE 3

LPL-Nimbi) is given in TABLE 1c. To verify that these are indeed P-positions, we could either check that the g-value of each position in TABLE 1c is 0, or use a process analogous to that used to prove that position number 19 is in P for LPL-Nimbi. To see that position number 9 is in P for LPW-Nimbi, verify that any move from it can be countered by a move to one of the positions 2, 4, 6, 7, 8, or 29. This completes (an outline of) the proof that LPW-Nimbi is a second-player winning game.

FIGURE 3 depicts a strategy graph for LPW-Nimbi, which is constructed analogously to the strategy graph of FIGURE 2. Position 9 is the source and position 20 is the terminal position. (Position 5 does not appear.) If the second player sticks to the strategy indicated by the graph, the first player can delay losing for at most ten moves, a possible sequence of moves being indicated by the P-positions $9 \to 8 \to 3 \to 23 \to 21 \to 20$.

For the interested reader we mention finally that LPL games are in general less tractable than LPW games. See, for example, Conway [**1**, Ch. 12] and Grundy and Smith [**5**]. Ferguson [**2**] found a subclass of tractable LPL games for which there is a winning strategy which is only a slight modification of the winning strategy of their LPW versions. It turns out that LPL-Nimbi is not in this class: Ferguson's condition A3 (which is also necessary) requires that if x is a component with g-value 1 and if y is a follower of x with g-value 0, then every component of y has g-value 0 or 1. The empty board (terminal position) has g-value 0. An isolated token has, therefore, g-value 1, a connected pair (like 37 or ED) has g-value 2. Also $g(137\text{TED})=0$, since position 28 in TABLE 1c has g-value 0 by property I. Since $g(1)=1$, the nim-sum yields $g(37\text{TED})=1$. Since the follower 37ED, whose components are 37 and ED, satisfies $g(37\text{ED})=$

$g(37) \oplus g(ED) = 2 \oplus 2 = 0$, condition A3 does not hold. We remark that even LPW-Nimbi, played on a board of arbitrary size, appears to be much harder than the classical LPW games, like those of Guy and Smith [6], because LPW-Nimbi is not a disjunctive sum of disjoint combinatorial games.

Acknowledgments

The *P*-positions of the LPW-version of a game similar to Nimbi, but played on a rectangular board and without the row contiguousness condition, were computed by R. B. Eggleton, A. S. Fraenkel and B. Rothschild in 1973 for all $2 \times n$ rectangles and $3 \times m$ $(m \leqslant 5)$ rectangles. They called the game 2-dimensional Nim, did not publish the results, and were unaware that D. Fremlin [4] had examined this game, which he called Nim-squared, at about the same time and, using a computer, computed all *P*-positions which fit into a 4×4 square for both the LPL and the LPW version. S. -Y. R. Li notified us that 2-dimensional Nim is also being played on a triangular board.

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Solving an Exponential Equation

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Gerald A. Heuer in [2] studied the equation

$$a^{x} = \left(\frac{a+x}{2}\right)^{(a+x)/2}, \qquad (1)$$

showing that for each a greater than e there is a unique solution with $x > a$. He gave numerical results, upper and lower estimates both asymptotic to $2a^2 - 2a \ln a - a$, and noted the integer solution $4^{12} = 8^8$. It turns out that this equation can be changed to the form

$$\frac{\ln u}{u} = \frac{\ln a}{a} \qquad (2)$$